Emergence et "Imergence" dans les systèmes complexes : application de l'agrégation de variables

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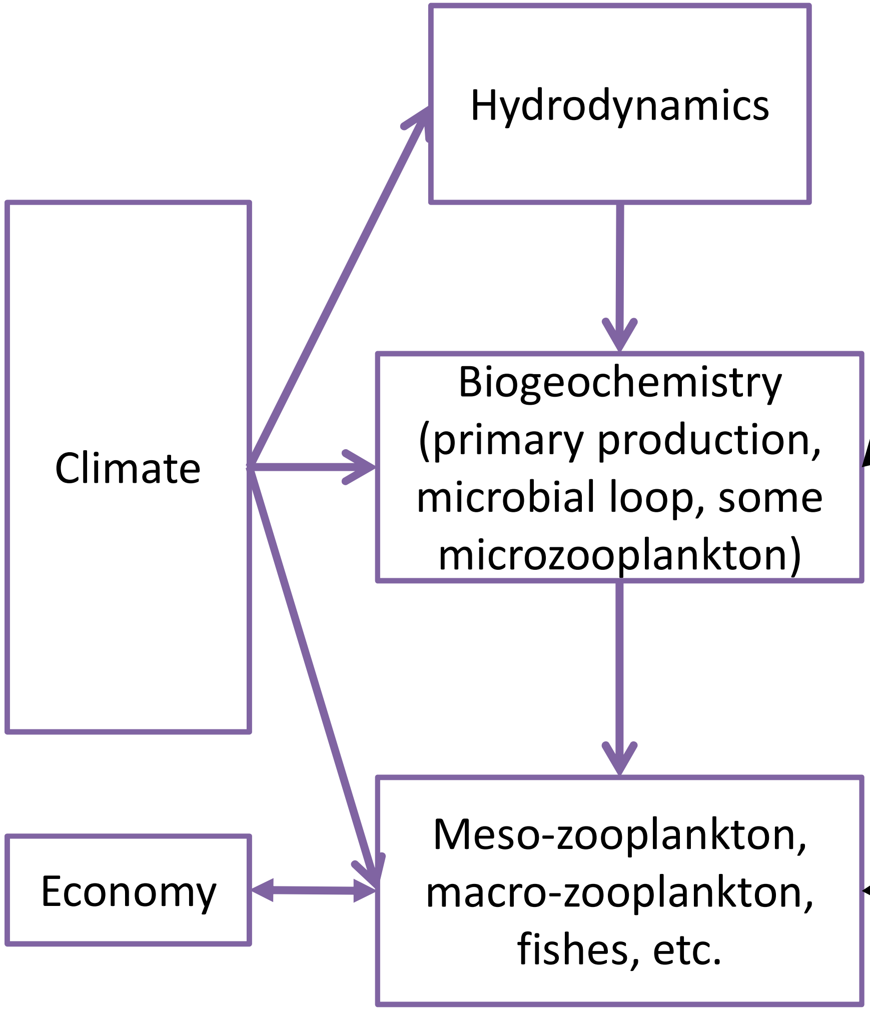
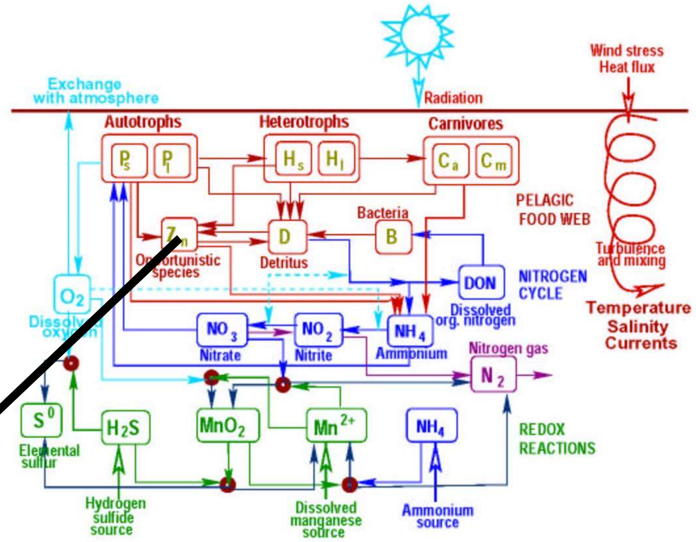
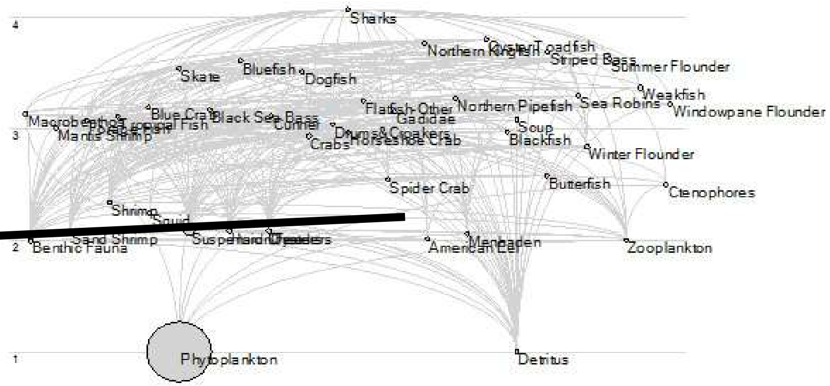
Introduction





http://web.mit.edu/rizzoli/Public/project2.html



http://www.somas.stonybrook.edu/~frisk/Nut.html

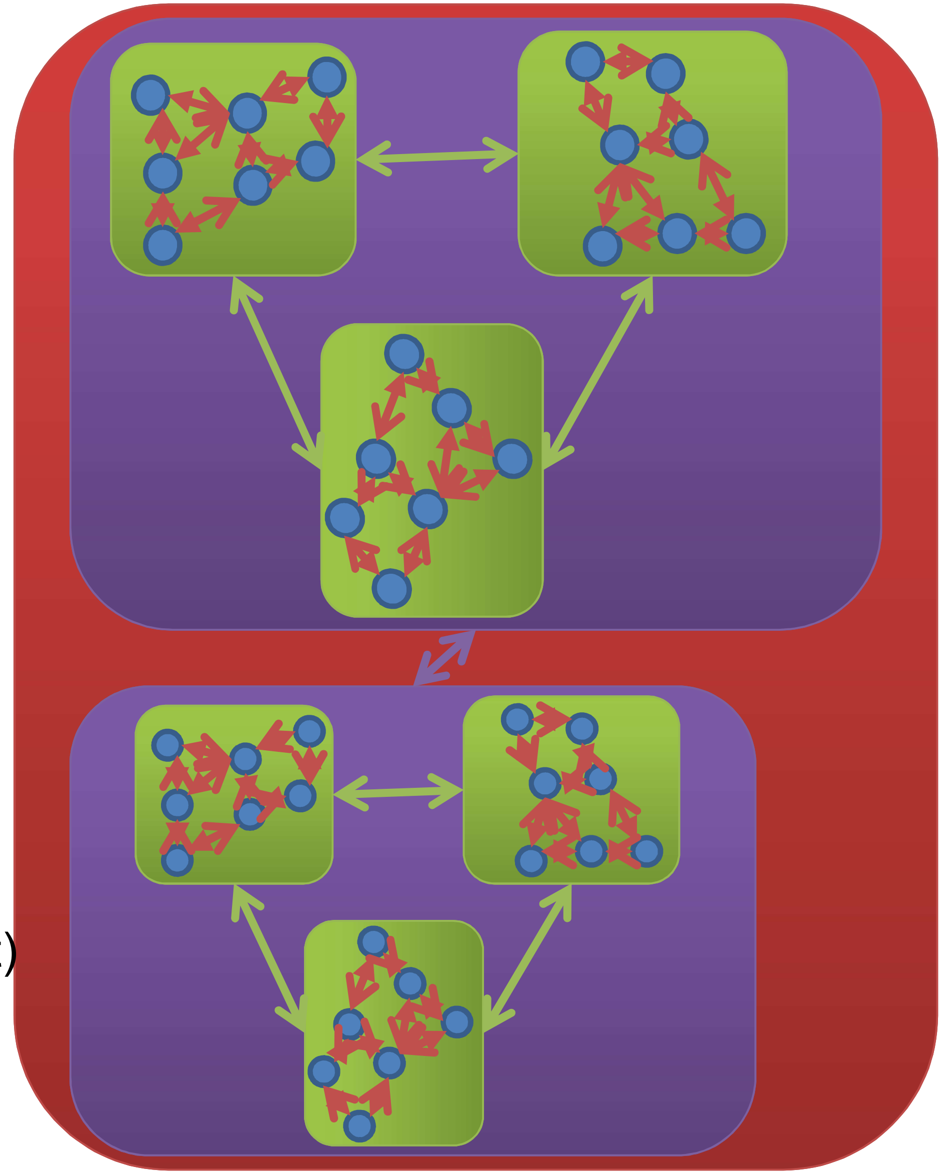
$$\frac{d\mathbf{n}}{d\tau} = F(\mathbf{n})$$

$$Y_j = \Phi_j(\mathbf{n})$$

dim(**Y**)<<dim(**n**)

n = micro – variables
Y = macro - variables

Hierarchical system



Colour	Organization level
(blue circle)	individuals
(green)	sub-population (*e.g.* same trait)
(purple)	population
(red)	community

Cas des systèmes impliquant plusieurs échelles de temps

- Dans de nombreux systèmes complexes, les processus opèrent à des **échelles de temps caractéristiques** propres qui ne sont pas toutes les mêmes.
- Si une variable d'état est gouvernée uniquement par des processus lents, on dit que c'est une **variable lente**.
- Si un processus rapide est impliqué dans la dynamique d'une variable d'état, celle-ci est une **variable rapide**.
- Une variable rapide peut être à l'**équilibre**, et dans ce cas elle ne varie pas.

Une macro-variable peut être définie comme une variable lente, c'est-à-dire qu'elle décrit la dynamique du système à long terme.

Systèmes lents - rapides

$$\frac{dx}{d\tau} = \boxed{F(x,y)} + \boxed{\varepsilon f(x,y)}$$
$$\frac{dy}{d\tau} = \boxed{\varepsilon G(x,y)}$$

RAPIDE

LENT

ε est un petit paramètre sans dimension appelé facteur d'échelles de temps.

I. $\varepsilon = 0$

$$\frac{dx}{d\tau} = F(x,y)$$
$$\frac{dy}{d\tau} = 0$$

y est constant (ne dépend pas du temps)

Systèmes lents - rapides

$$\frac{dx}{d\tau} = F(x,y) + \varepsilon f(x,y)$$ **RAPIDE**

$$\frac{dy}{d\tau} = \varepsilon G(x,y)$$ **LENT**

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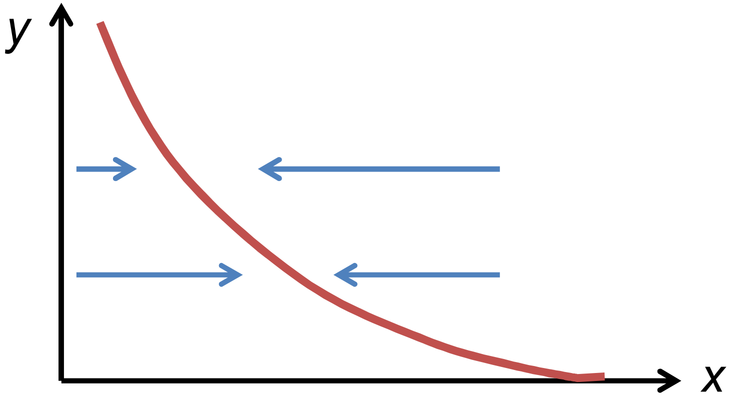
Hypothèse : Supposons que pour tout *y*, *x* tend vers un équilibre noté *x*(y)*.



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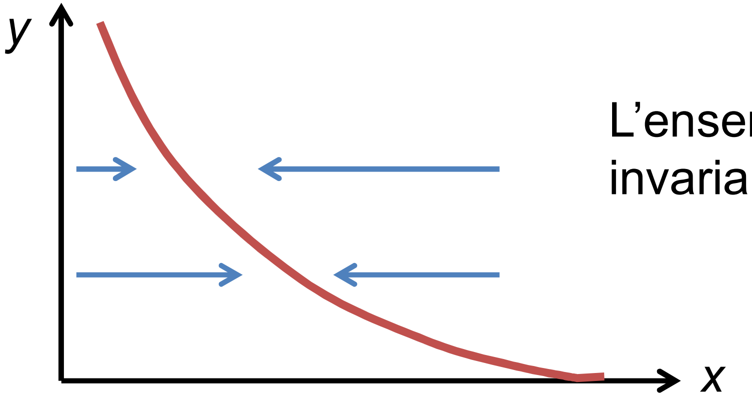

L'ensemble des équilibres est une variété invariante.

Systèmes lents - rapides

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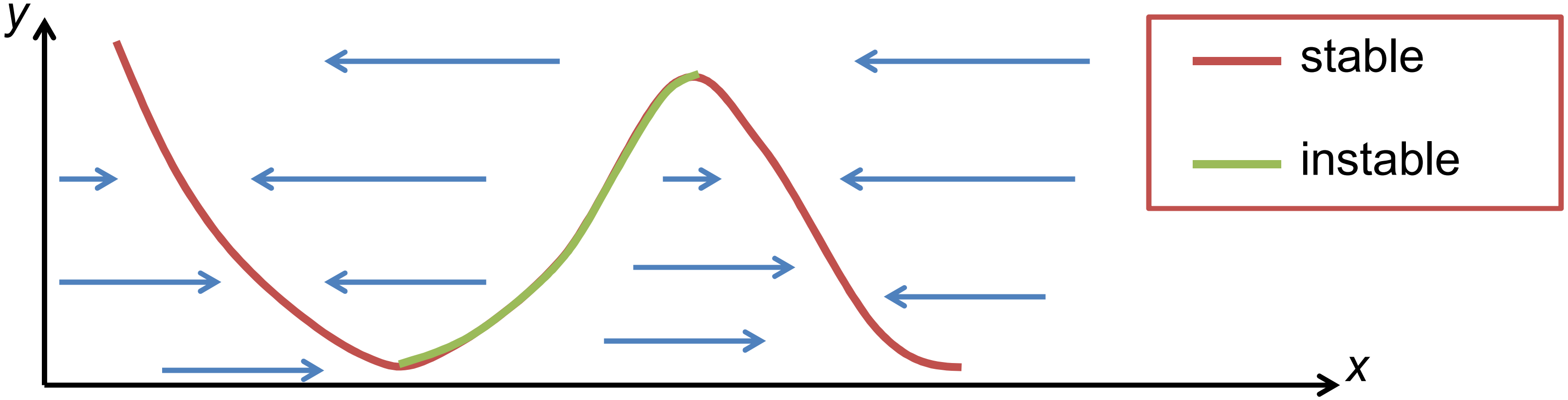

Systèmes lents - rapides

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$$\frac{dy}{d\tau} = \boxed{\varepsilon G(x,y)}$$ **LENT**

ε est un petit paramètre sans dimension appelé facteur d'échelles de temps.

II - $\varepsilon \neq 0$ Recette : Remplacer *x* by x*(y) dans l'équation de y et changer le temps

$$t = \varepsilon \tau$$

$$\frac{dy}{dt} = G(x^*(y), y) = \tilde{G}(y)$$ Modèle agrégé (MA)

$$x = x^*(y) \longrightarrow$$ Fournit les micro-variables si on connait les macro-variables avec le MA.

<u>**Théorème de Tychonov**</u> : les solutions du MA sont des « bonnes » approximations des solutions du modèle complet pour un temps « long » (de l'ordre de $1/\varepsilon$)

Systèmes lents - rapides

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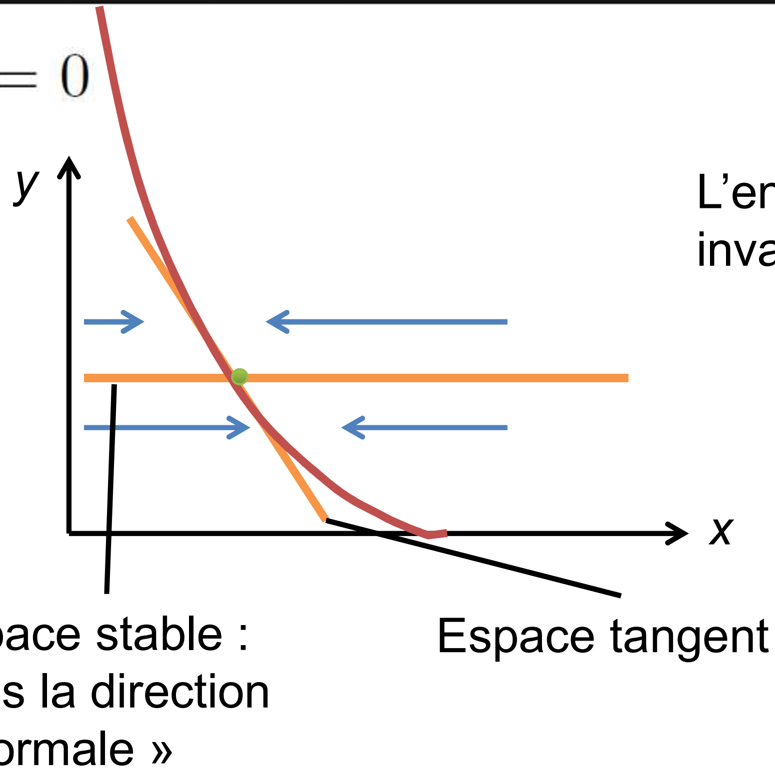
Quasi-steady state approximation, adiabatic assumption, time-scale separation

Idée intuitive … mais parfois insuffisante.

Systèmes lents - rapides

I - $\varepsilon = 0$



L'ensemble des équilibres est une variété invariante, on la note $\mathcal{M}_0$

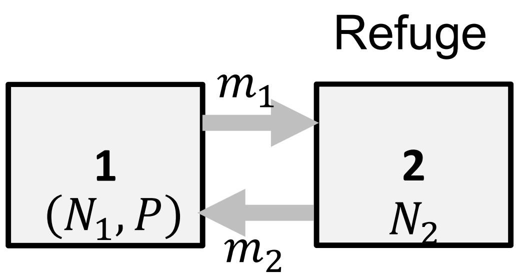
L'ensemble des équilibres est une variété normalement hyperbolique.

Wiggins S. Normally hyperbolic invariant manifolds in dynamical systems. AMS, vol. 105. New York: Springer-Verlag; 1994.

$$Théorème\ (Fenichel, 1971, 1979)$$

$$si\ \varepsilon\ est\ assez\ petit, la\ variété\ invariante\ \mathcal{M}_0\ persiste\ dans\ le\ système\ perturbé$$

Exemple

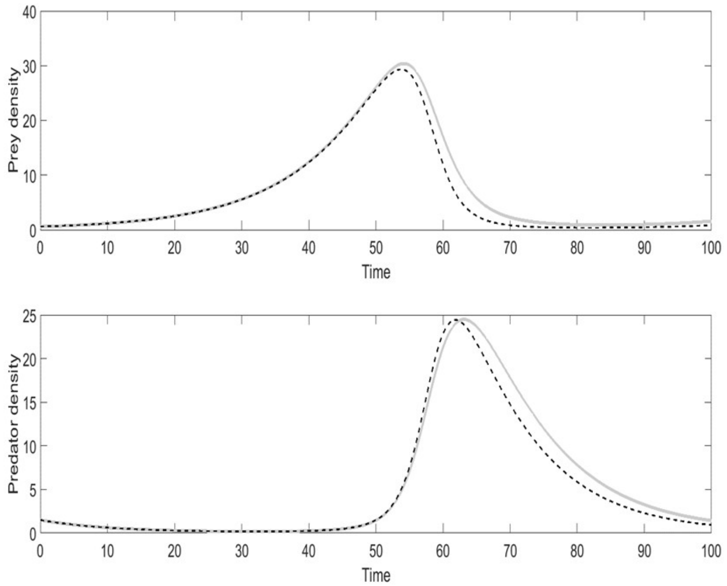


$$\frac{dx_1}{d\tau} = m_2 x_2 - m_1 x_1 + \varepsilon x_1 (r_1 - ay)$$
$$\frac{dx_2}{d\tau} = m_1 x_1 - m_2 x_2 + \varepsilon x_2 r_2$$
$$\frac{dy}{d\tau} = \varepsilon y (b x_1 - d)$$

$$u_1^* = \frac{m_2}{m_1 + m_2} \quad \text{and} \quad u_2^* = \frac{m_1}{m_1 + m_2}$$

$$\frac{dx}{dt} = x(r - a_1 y)$$
$$\frac{dy}{dt} = y(b_1 x - d)$$

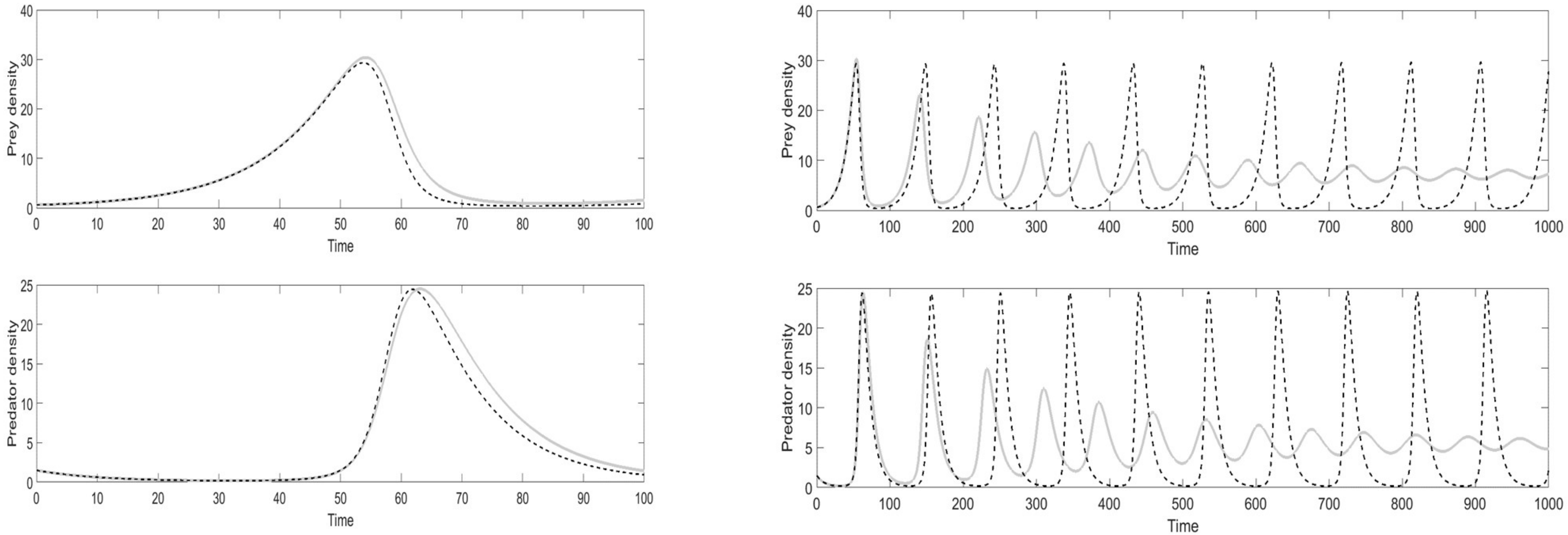
Exemple



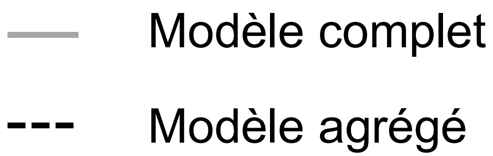
Modèle complet

--- Modèle agrégé

Exemple

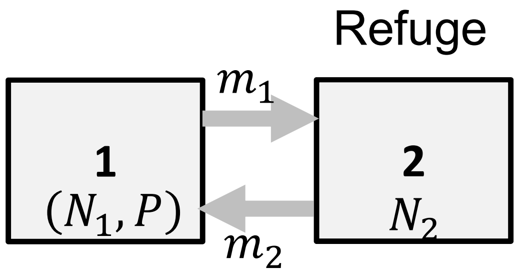


Prey density
Time
Predator density
Time
Prey density
Time
Predator density
Time



Modèle complet
Modèle agrégé

Exemple



$$\frac{dx_1}{d\tau} = m_2 x_2 - m_1 x_1 + \varepsilon x_1 (r_1 - a y)$$
$$\frac{dx_2}{d\tau} = m_1 x_1 - m_2 x_2 + \varepsilon x_2 r_2$$
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$$u_1^* = \frac{m_2}{m_1 + m_2} \quad \text{and} \quad u_2^* = \frac{m_1}{m_1 + m_2}$$

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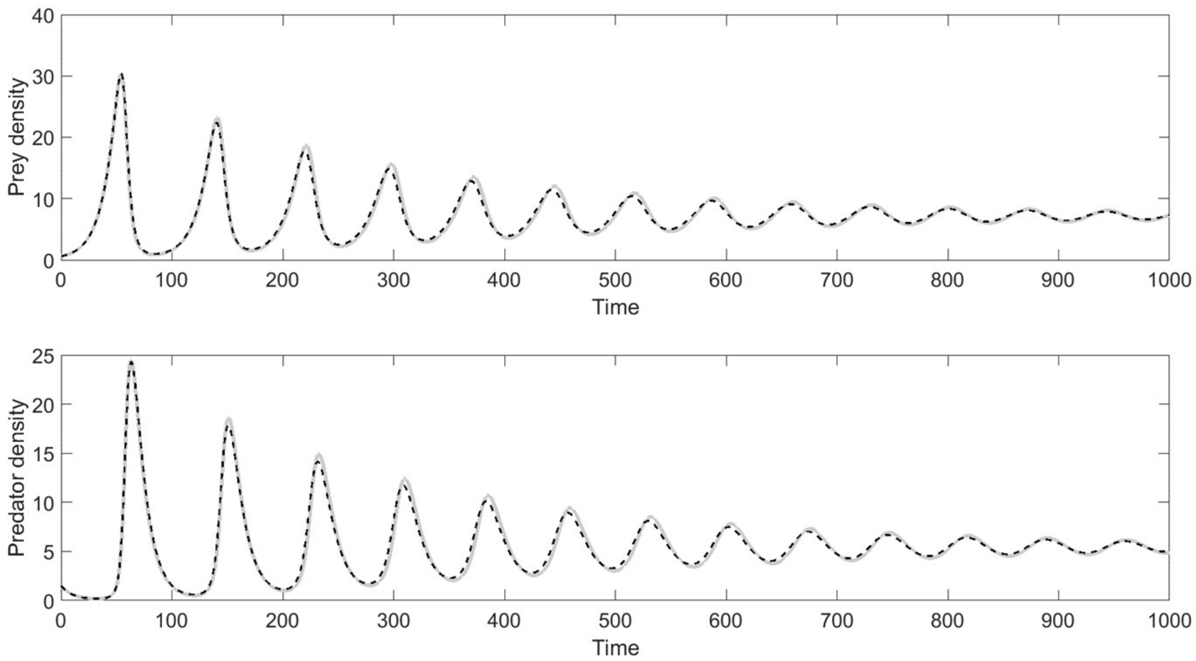
Exemple



Remarque : une étude du modèle agrégé permet de démontrer la stabilité de l'équilibre et l'effet stabilisant de la structure spatiale avec refuge dans le système prédateur – proie.

J.-C. Poggiale , P. Auger, (2004), Impact of spatial heterogeneity on a predator–prey system dynamics, C. R. Biologies, 327, 1058–1063

Emergence (émergence fonctionnelle)

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{K}\right) - \frac{ax}{1+bx}y$$

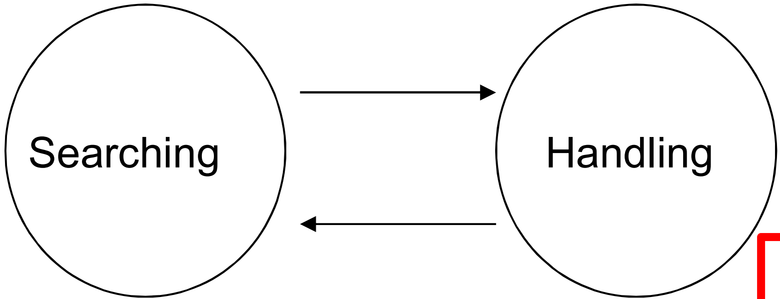
$$\frac{dy}{d\tau} = \varepsilon y\left(e_1 \frac{ax}{1+bx} - \mu_1\right)$$

Idée de Holling:

$$\Delta t = \Delta t_s + \Delta t_h$$

$$\Delta x = ax\Delta t_s$$

$$\Delta t_h = T_h \Delta x = T_h ax \Delta t_s$$

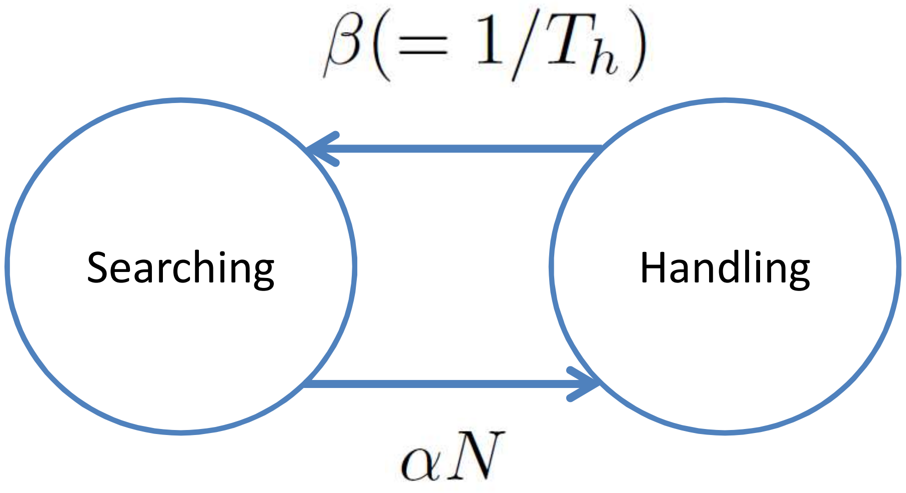


$$\frac{\Delta x}{\Delta t} = \frac{ax\Delta t_s}{\Delta t_s + T_h ax \Delta t_s} = \frac{ax}{1 + T_h ax}$$

C.S. Holling, Some characteristics of simple types of predation and parasitism, *Can. Ent.* 91, 385–398 (1959).

Emergence (émergence fonctionnelle)

Emergence d'une formulation de réponse fonctionnelle à long termes sur la base d'hypothèses sur les processus rapides.



N Numbre de proies

P_h Nombre de prédateur manipulant une proie

P_s Nombre de prédateur à la recherche d'une proie

Réponse fonctionnelle $g(N) = aNP_s/P$

Quelle est la réponse fonctionnelle à long terme?

$$\frac{dP_s}{d\tau} = -\alpha N P_s + \beta P_h + \varepsilon(\text{demography})$$

$$\frac{dP_h}{d\tau} = \alpha N P_s - \beta P_h + \varepsilon(\text{demography})$$

$$P = P_s + P_h$$

Emergence (émergence fonctionnelle)

$$-\alpha N P_s + \beta P_h = 0$$

$$-\alpha N P_s + \beta P - \beta P_s = 0$$

$$P_s = \frac{\beta}{\beta + \alpha N} P$$

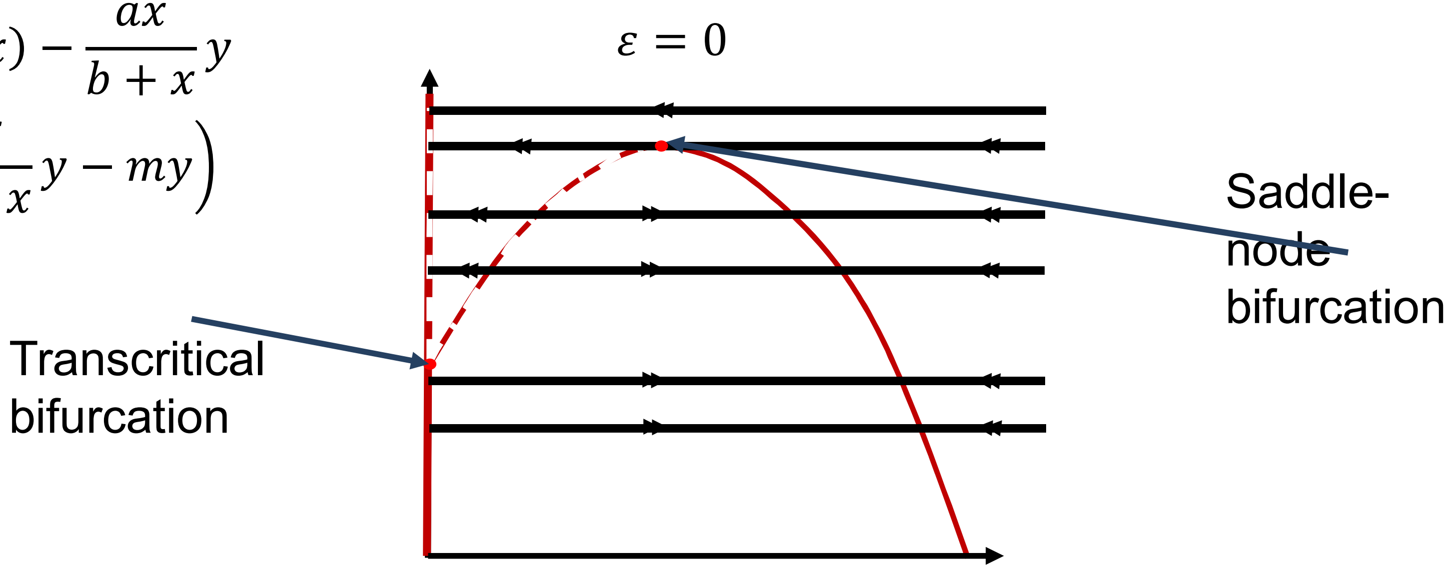
$$g(N) = a N P_s / P$$

$$g(N) = \frac{a\beta N}{\beta + \alpha N} = \frac{aN}{1 + \alpha T_h N}$$

Imergence : rétroaction des variables lentes sur la dynamique rapide

Lorsque la dynamique lente opère, elle peut pousser le système vers des états où la dynamique rapide subit une bifurcation.

$$\begin{cases} \dfrac{dx}{d\tau} = x(1-x) - \dfrac{ax}{b+x}y \\ \dfrac{dy}{d\tau} = \varepsilon\left(\dfrac{ax}{b+x}y - my\right) \end{cases}$$



- Dumortier F. and Roussarie R. (1996) Canard cycles and Center Manifolds. Memoir. Am. Math. Soc., 121, 1-100.
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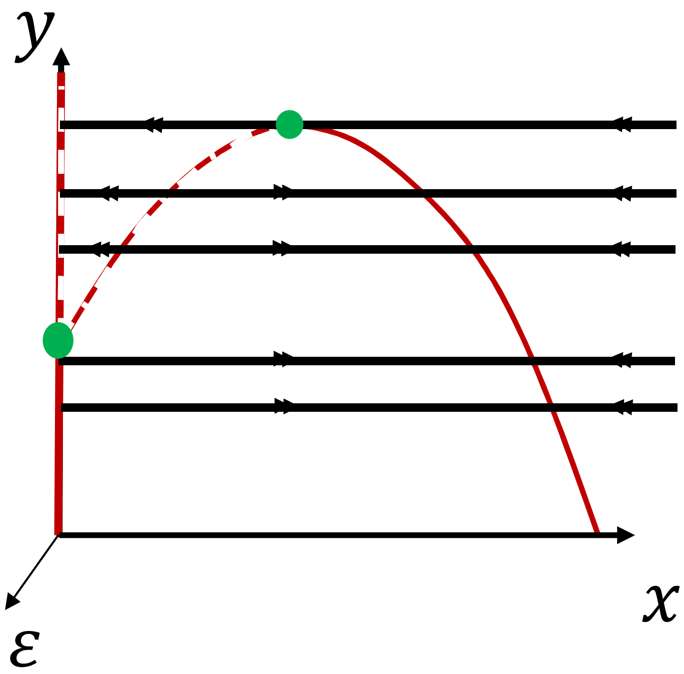
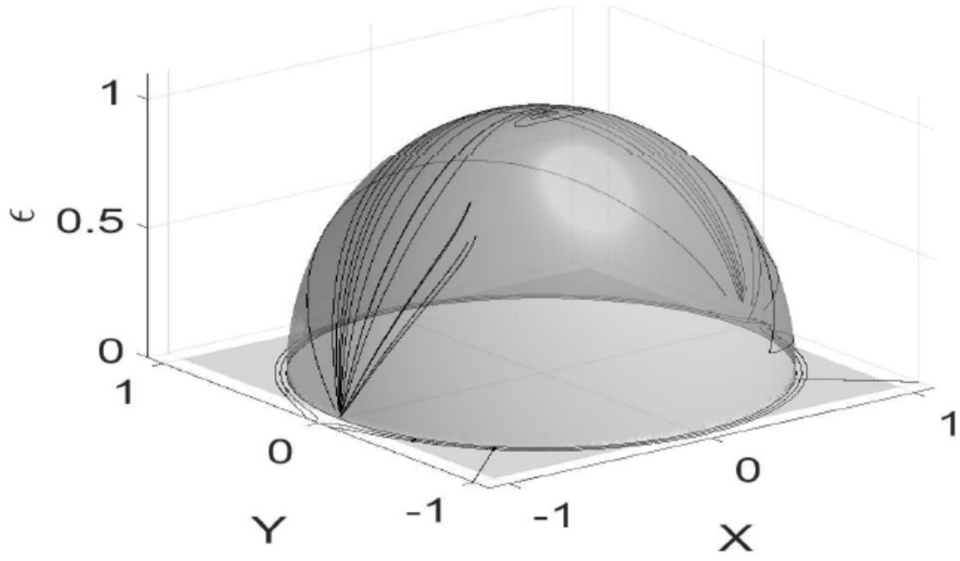
- Krupa, M., Szmolyan, P., (2001) Extending Geometric Singular Perturbation Theory to Nonhyperbolic Points---Fold and Canard Points in Two Dimensions, *SIAM J. Math. Anal.*, 33(2), 286–314.
- Vidal, A., Françoise, J.P., (2012), Canards cycles in global dynamics, Int. J. Bif. Chaos, 22.

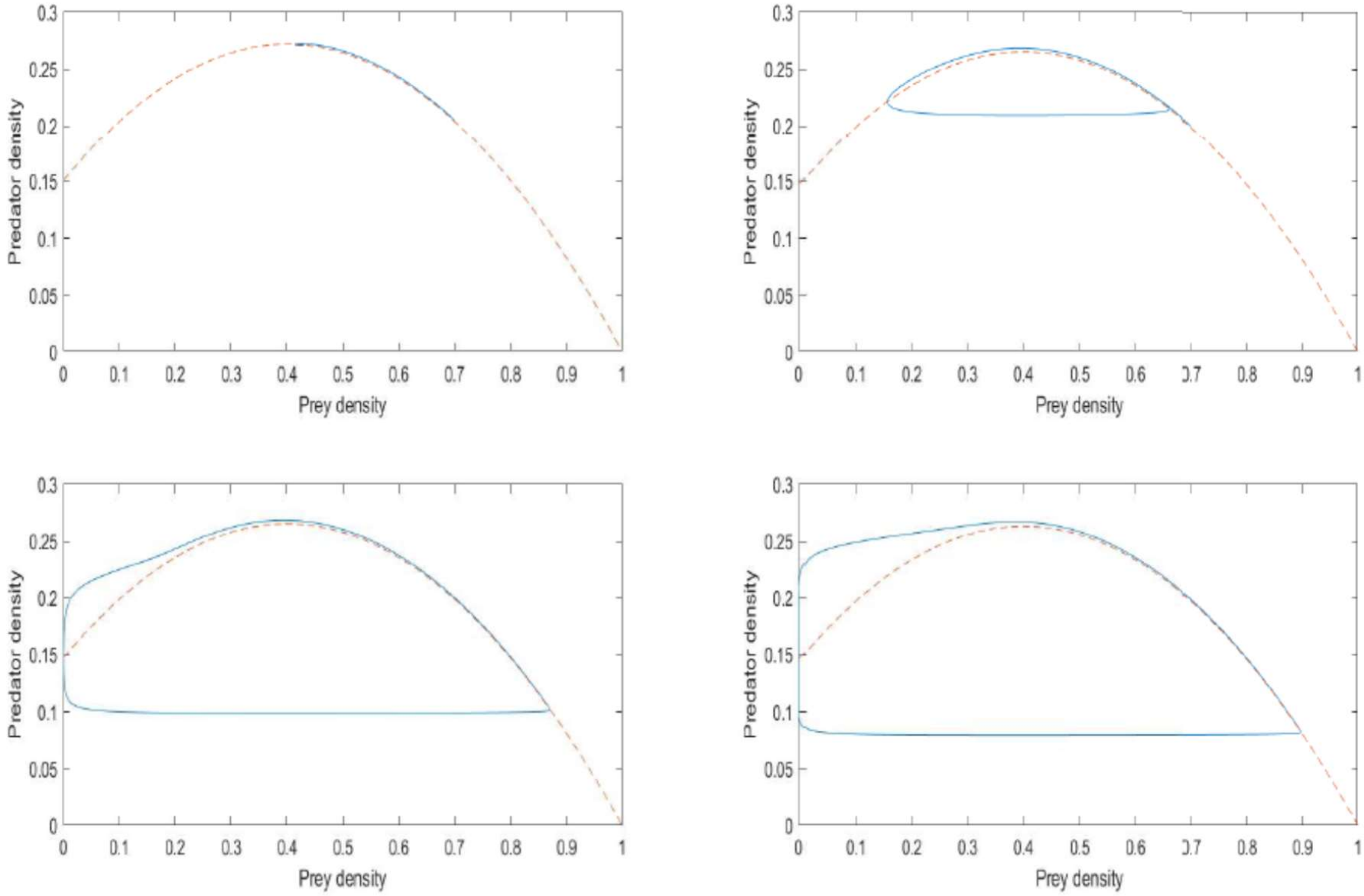
Imergence : rétroaction des variables lentes sur la dynamique rapide

Désingularisation : éclatement

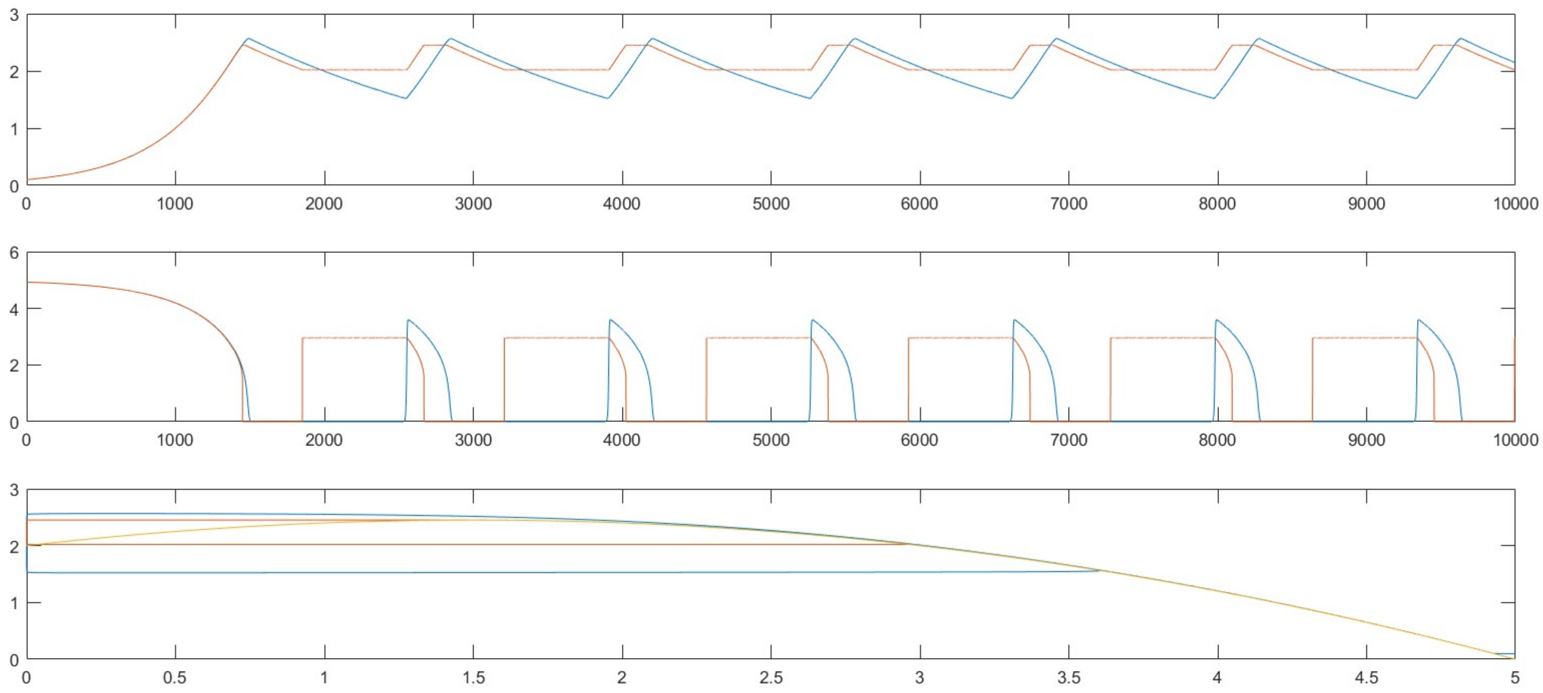




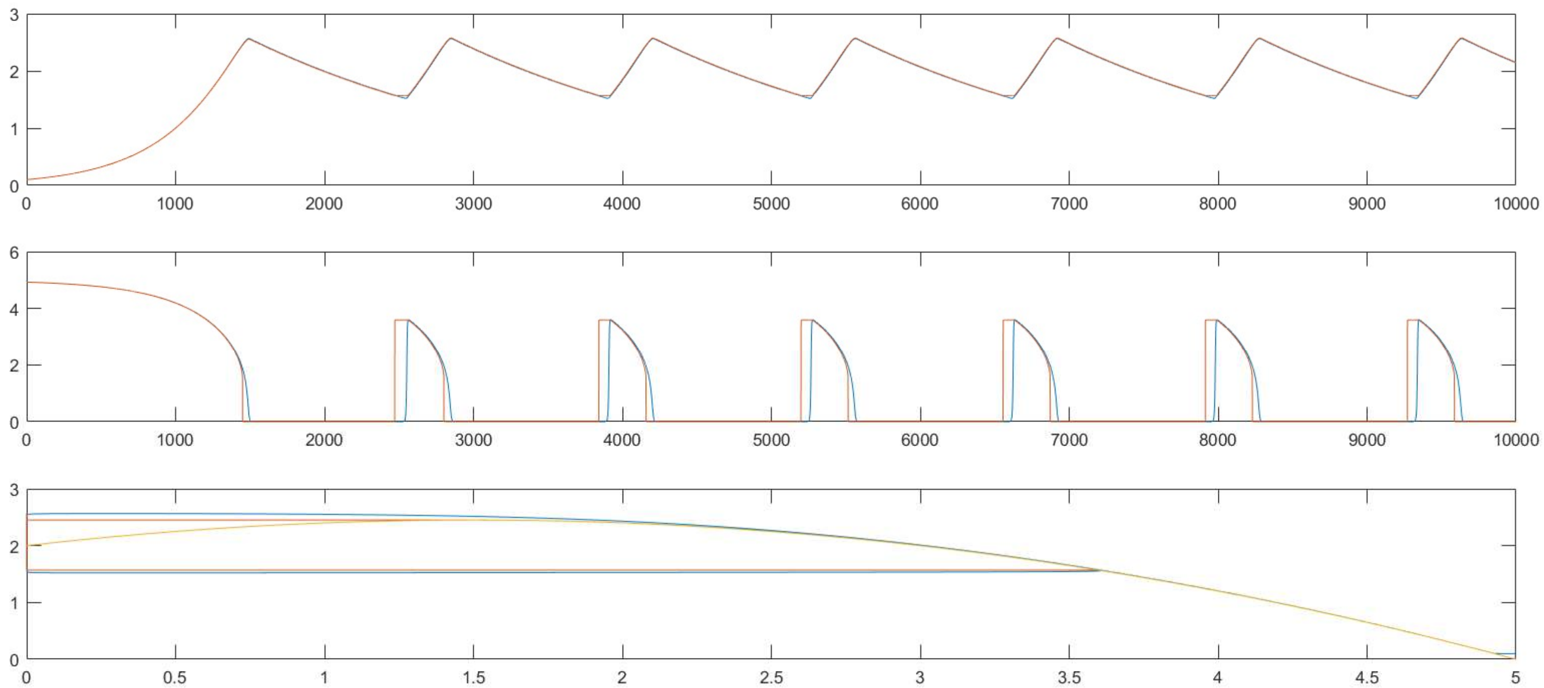
Imergence : rétroaction des variables lentes sur la dynamique rapide



Imergence : rétroaction des variables lentes sur la dynamique rapide



Imergence : rétroaction des variables lentes sur la dynamique rapide



Conclusions

Simplification de modèles sans perdre la complexité (pas d'approximation) : facilite l'étude mathématique

Réduction de dimension : peut permettre d'accélerer les temps de calcul

Lien entre les différents niveaux d'organisation : développements théoriques en écologie

Formulation mathématique de processus à « grandes » échelles sur la base de connaissances à « petites » échelles (mécanismes)

Mais :

Comment mettre un système quelconque avec des échelles de temps différentes sous la forme d'un système lent-rapide?

Comment trouver les macro-variables? (conservatives à « courte » échelle)

Merci pour votre attention

Pierre AUGER

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Tri NGUYEN HUU

Robert ROUSSARIE

Eva SANCHEZ

Blowing-up method : application to the fold point in the RMA model

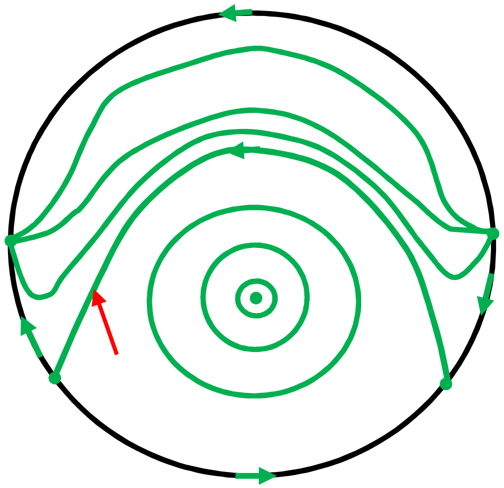
In the chart $\varepsilon = 1$:

$$\begin{cases} \dfrac{dx}{d\tau} = r\dfrac{1-b}{2}\left(\lambda x - x^2 - m\dfrac{1+b}{1-b}y\right) + O(r^2) \\ \dfrac{dy}{d\tau} = r\dfrac{(1+b)b}{2}x + O(r^2) \end{cases}$$

For $\lambda = 0$, after division by r, on the sphere, one gets:

$$\begin{cases} \dfrac{dx}{d\tau} = \dfrac{1-b}{2}\left(-x^2 - m\dfrac{1+b}{1-b}y\right) \\ \dfrac{dy}{d\tau} = \dfrac{(1+b)b}{2}x \end{cases}$$

$$y = -\frac{1-b}{m(1+b)}x^2 + \frac{b(1+b)}{2(1-b)}$$



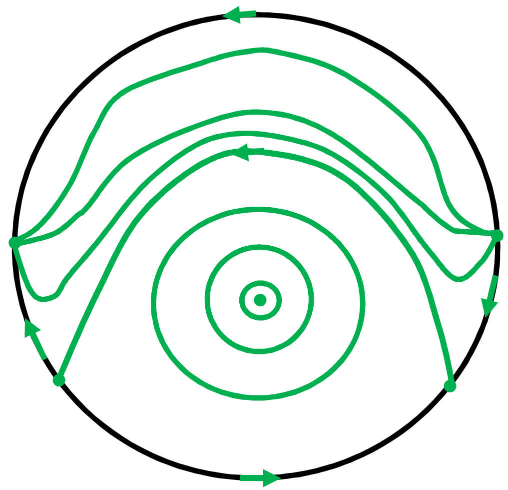
The parabola $y = -\frac{1-b}{m(1+b)}x^2 + \frac{b(1+b)}{2(1-b)}$ is invariant under the flow, it is a separatrix.

Blowing-up method : application to the fold point in the RMA model

$$\begin{cases} \dfrac{dx}{d\tau} = \dfrac{1-b}{2}\left(-x^2 - m\dfrac{1+b}{1-b}y\right) \\ \dfrac{dy}{d\tau} = \dfrac{(1+b)b}{2}x \end{cases}$$

Let the function H defined by:

$$H(x,y) = \left(x^2 + m\frac{m(1+b)}{1-b}y - mb\frac{(1+b)^2}{2(1-b)^2}\right)e^{\frac{2(1-b)}{b(1+b)}y}$$



$H \geq 0$ under the parabola and is a first integral.

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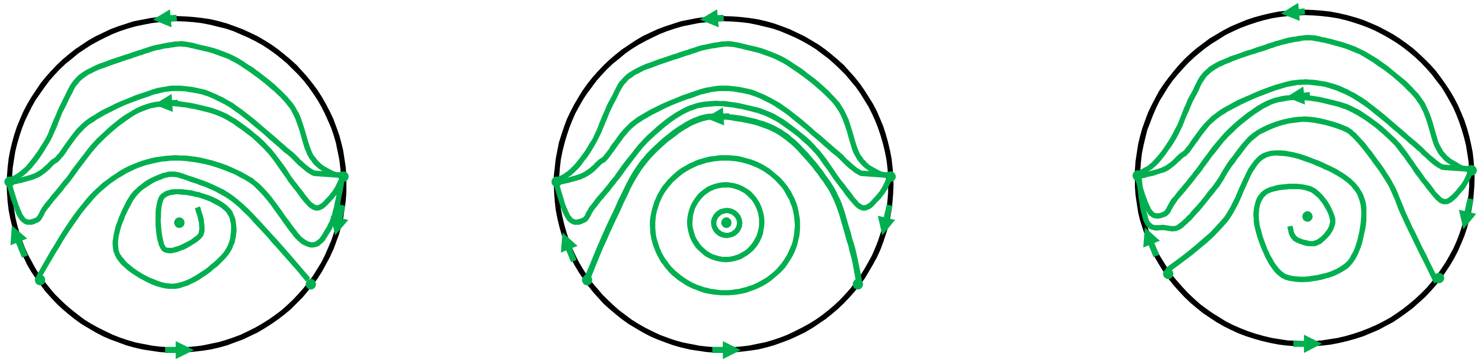
By using a Melnikov integral for perturbation of centres, we can prove that, there is no limit cycle around $(0,0)$ when $\lambda \neq 0$ is close to 0.

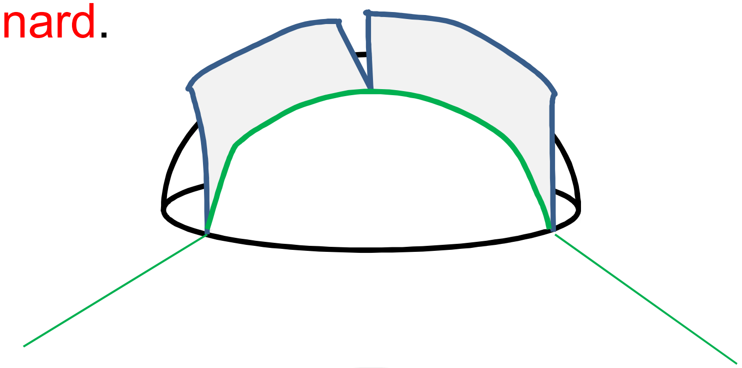


$\lambda < 0$ $\lambda = 0$ $\lambda > 0$

Blowing-up method : application to the fold point in the RMA model

Notations: $\mathcal{M}_0$, the critical manifold (parabola) has a stable branch denoted $\mathcal{M}_0^S$ and an unstable branch $\mathcal{M}_0^U$. Now, for ε small and positive, the normally hyberbolic stable branch persists in $\mathcal{M}_\varepsilon^S$ and the normally hyberbolic unstable branch persists in $\mathcal{M}_\varepsilon^U$.

Definition: a solution lying in the intersection of $\mathcal{M}_\varepsilon^S$ and $\mathcal{M}_\varepsilon^U$ is called a maximal canard.



Blowing-up method : application to the fold point in the RMA model

In the chart $\varepsilon = 1$, the vector field can be written as:

$$F = F_{00} + \lambda F_{1\lambda} + rF_{1r} + O(r^2) + O(\lambda^2)$$

Where:

$$F_{00} = \begin{pmatrix} \frac{1-b}{2}\left(-x^2 - m\frac{1+b}{1-b}y\right) \\ \frac{b(1+b)}{2}x \end{pmatrix}, \quad F_{1\lambda} = \begin{pmatrix} \frac{1-b}{2}\lambda x \\ 0 \end{pmatrix}$$

and

$$F_{1r} = \begin{pmatrix} -\frac{\lambda^2}{2}x + \frac{\lambda^2}{2}x^2 - m\frac{1+b}{1-b}xy + m\frac{1+b-b\lambda}{1-b}y \\ \frac{b\lambda}{2}x \end{pmatrix}$$

Blowing-up method : application to the fold point in the RMA model

$$F_{00} = \begin{pmatrix} \frac{1-b}{2}\left(-x^2 - m\frac{1+b}{1-b}y\right) \\ \frac{b(1+b)}{2}x \end{pmatrix} = \begin{pmatrix} R\frac{\partial H}{\partial y} \\ -R\frac{\partial H}{\partial x} \end{pmatrix}$$

$$\Delta H = \int_0^t \frac{dH}{ds}\big(\Gamma_c(s)\big)ds = \int_0^t \nabla H\big(\Gamma_c(s)\big).F\big(\Gamma_c(s)\big)ds$$

$$= \underbrace{\int_0^t \nabla H\big(\Gamma_c(s)\big).F_{00}\big(\Gamma_c(s)\big)ds}_{=0} + \underbrace{r\int_0^t \nabla H\big(\Gamma_c(s)\big).F_{1r}\big(\Gamma_c(s)\big)ds}_{=r\alpha_r} + \underbrace{\lambda\int_0^t \nabla H\big(\Gamma_c(s)\big).F_{1\lambda}\big(\Gamma_c(s}_{=\lambda\alpha_\lambda}$$

Blowing-up method : application to the fold point in the RMA model

From the implicit function theorem, in a neighborhood of $(r, \lambda) = (0,0)$, there exists a function λ_c of r such that $\Delta H = 0$ is equivalent to $\lambda = \lambda_c(r)$ and one gets:

$$\lambda_c(r) = -\frac{\alpha_r}{\alpha_\lambda} r + O(r^2)$$

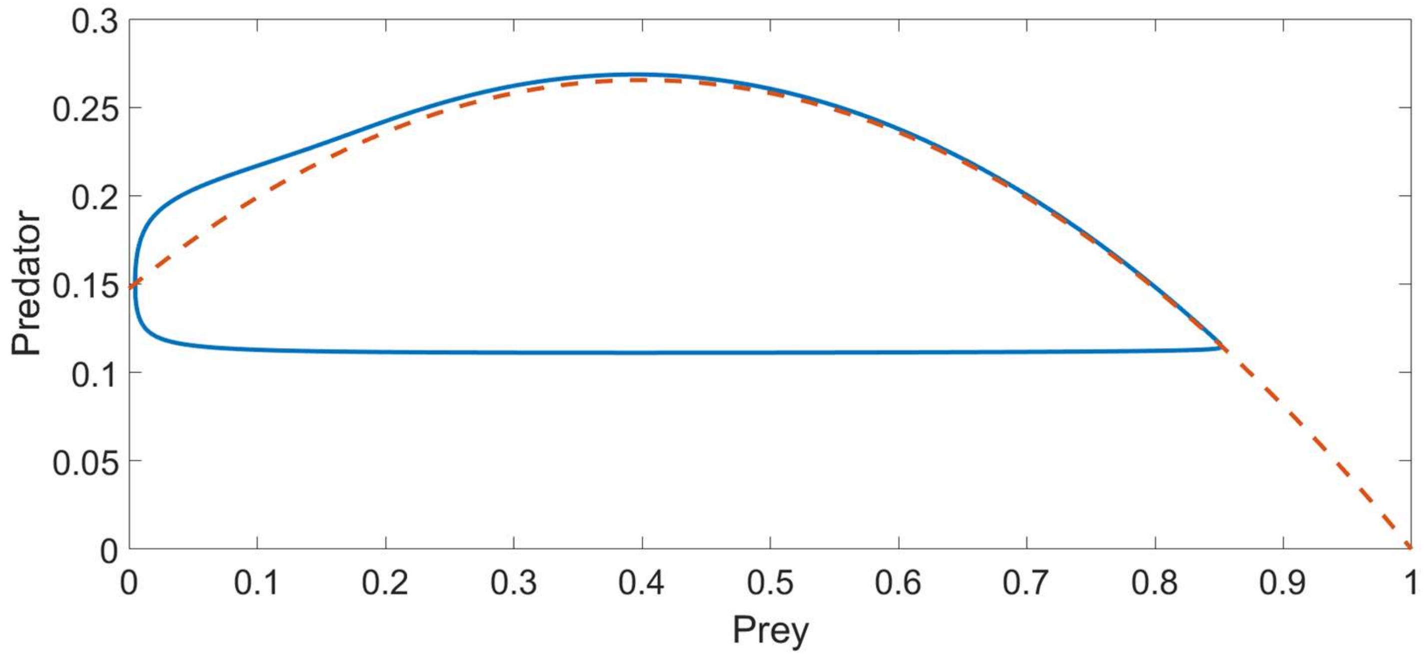
One can show that:

$$\lambda_c(r) = \frac{4(1-b)}{mb(1+b)^2} r + O(r^2)$$

And since we have $r^2 = \varepsilon$, the previous equation can be written as follows:

$$\lambda_c(\varepsilon) = \frac{4(1-b)}{mb(1+b)^2} \sqrt{\varepsilon} + O(\varepsilon)$$

Blowing-up method : application to the fold point in the RMA model



Conclusion

- Same approach done for the transcritical bifurcation (intersection between parabola and vertical axis)

- Ronsenzweig Mc Arthur model, a simple example with interesting properties. Comparison with « mass balanced » models (B. Kooi);

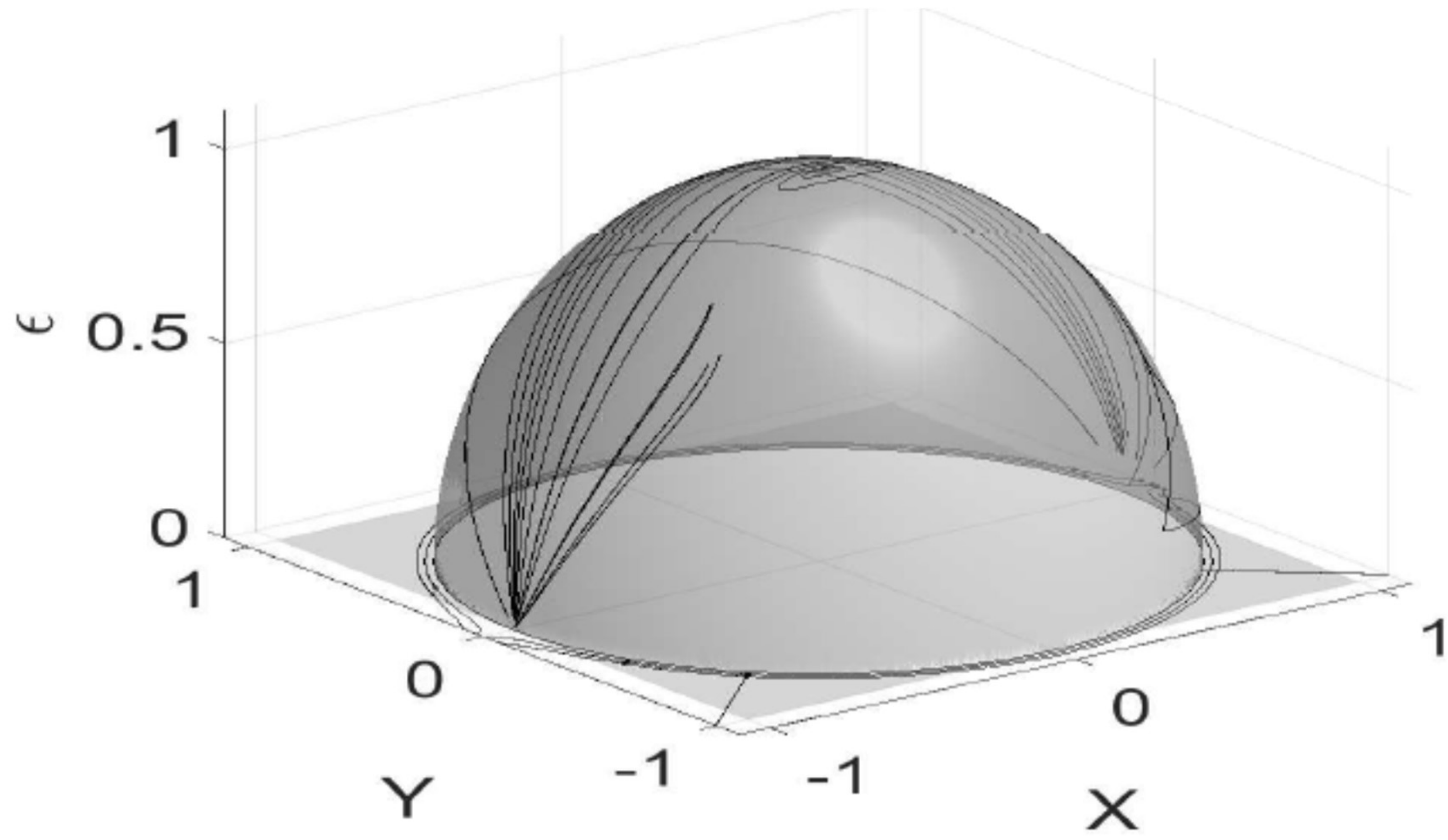
- Geometrical Singular Perturbation theory provides simple tools to analyze multiple time scales models;

- Extensions like blowup allow to deal with non normal hyperbolicity;

- Can be used in higher dimension (generic canards);

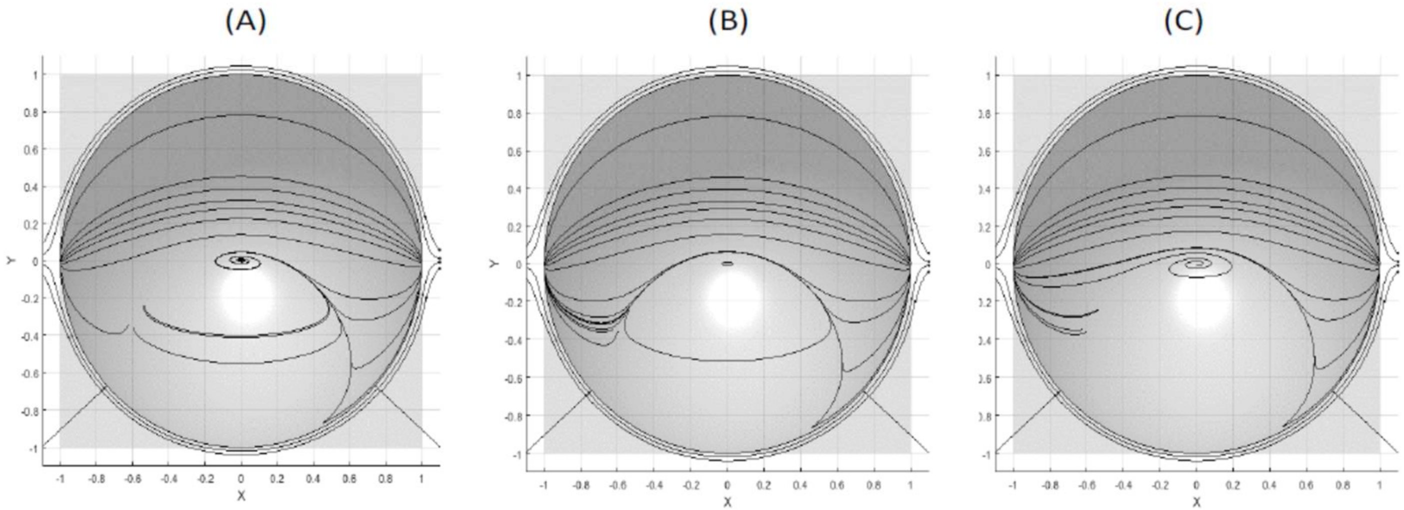
- simple calculations for complex phenomenons :
    - delayed bifurcation
    - mixed-mode and burst oscillations in multiple time scales;

- Modeling: Mechanisms including several time scales allow to use asymptotics in



ϵ
1
0.5
0
1
0
-1
Y
-1
0
1
X

$$\frac{dX_1}{dt} = \frac{1-b}{2}\left(\lambda_1 X_1 - m\frac{1+b}{1-b}Y_1 - X_1^2\right) + O(r)$$
$$\frac{dY_1}{dt} = \frac{(1+b)b}{2}X_1 + O(r)$$



$$H(X_1, Y_1) = \exp\left(\frac{2(1-b)}{b(1+b)} Y_1\right)\Big(X_1^2 + \frac{m(1+b)}{1-b} Y_1 - \frac{mb(1+b)^2}{2(1-b)^2}\Big)$$

$$\delta(r, \lambda_1) = \int_{-\infty}^{+\infty} dH(\gamma(t)) = \int_{-\infty}^{+\infty} \frac{dH(\gamma(t))}{dt} dt$$

$$\delta(r, \lambda_1) = \alpha_r r + \alpha_{\lambda_1} \lambda_1 + O(r(r + \lambda_1))$$

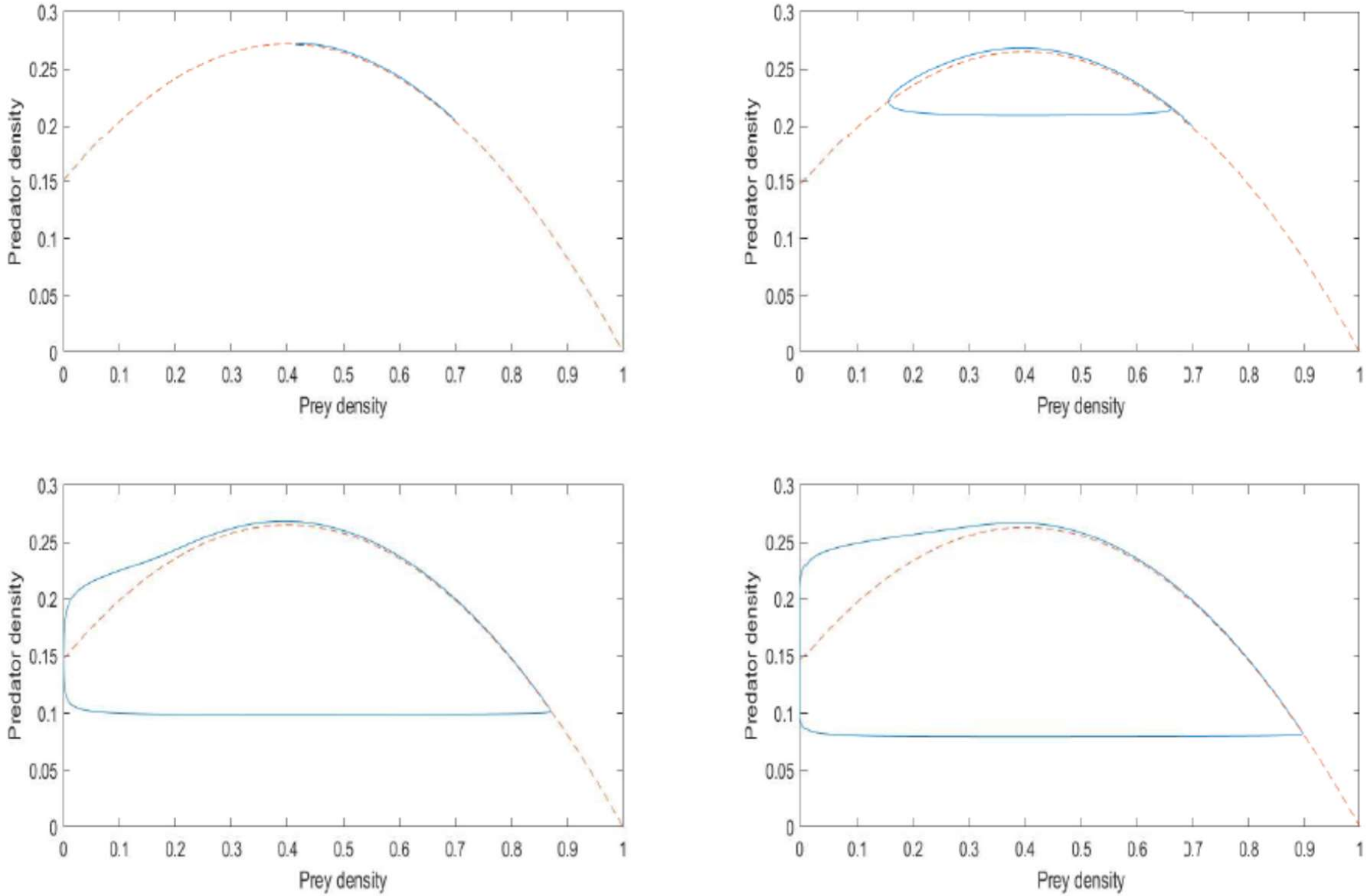
$$\alpha_r = \int_{-\infty}^{+\infty} \nabla H(\gamma(t)).F_{1,r}(\gamma(t))dt$$

$$\alpha_{\lambda_1} = \int_{-\infty}^{+\infty} \nabla H(\gamma(t)).F_{1,\lambda_1}(\gamma(t))dt$$

$$\lambda_{1c}(r) = \frac{mb(1+b)^2}{(1-b)^3} r + O(r^2)$$

$r = \sqrt{\varepsilon} \quad \lambda = \sqrt{\varepsilon}\lambda_1$

$$\lambda_c(\sqrt{\varepsilon}) = \frac{mb(1+b)^2}{(1-b)^3} \varepsilon + O(\varepsilon^{3/2})$$



0.3
0.25
0.2
0.15
0.1
0.05
0
Predator density
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Prey density
0.3
0.25
0.2
0.15
0.1
0.05
0
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0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
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Prey density

Conclusions (1/2)

Time scale separation: existence of a invariant subset in the phase space, with a lower dimension that the whole space, on which the dynamics can be reduced.

Recipe : Consider fast variables at equilibrium, neglecting slow dynamics, and replace these fast variables by their equilibrium.

This provides the first order approximation of the invariant subset : if the reduction of the dynamics to this approximate subset is not structurally stable, need to determine higher order terms in the approximation.

The reduction is valid if the subset is « normally hyperbolic ». In dynamical systems, long term dynamics can push the system away from the region where the subset is normally hyperbolic : loss of normal hyperbolicity.

Conclusions (1/2)

Simplification of models : easier mathematical study

Reduction of dimension : faster numerical simulations

Link between different organization level : theoretical development in ecology

Mathematical formulation of processes at large scales based on knowledge at small scales (or vice-versa)

Conclusions (1/2)

Simplification of models : easier mathematical study

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Mathematical formulation of processes at large scales based on knowledge at small scales (or vice-versa)

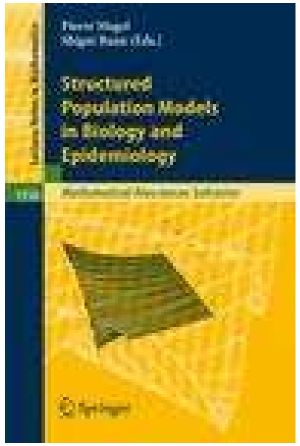
BUT :

How to transform a system with slow and fast processes in its slow-fast form?

How to find a macro-variable? (conservative at slow time scale)

Extensions

P. Auger, R. Bravo de la Parra, J.-C. Poggiale, E. Sanchez, and T. Nguyen-Huu, 2008, **« Aggregation of Variables and Applications to Population Dynamics »** *in* Structured Population Models in Biology and Epidemiology
Series: Lecture Notes in Mathematics
Subseries: Mathematical Biosciences Subseries , Vol. 1936
Magal, Pierre; Ruan, Shigui (Eds.) , 345 p.



Discrete systems

Delayed and partial differential equations

Non autonomous systems

Applications to population dynamics models

Thanks for your attention

•M. Marva, J.-C. Poggiale, R. Bravo de la Parra, 2013, Reduction of slow-fast periodic systems with applications to population dynamics models, *Mathematical Models and Methods in Applied Sciences*, 22 (10).

•P. Auger, J.-C. Poggiale, E. Sanchez, 2012, A review on spatial aggregation methods involving several time scales, *Ecological Complexity* 10, 12-25

•M. Marva, R. Bravo de la Parra, J.-C. Poggiale, 2012, Approximate aggregation of a two time scales periodic multi-strain SIS epidemic model: a patchy environment with fast migrations, *Ecological Complexity* 10, 34-41

•E. Sánchez, P. Auger, J.-C. Poggiale, 2012, Two-time scales in spatially structured models of population dynamics: A semigroup approach, *Journal of Mathematical Analysis and Applications*, 375, 149-165

•J.-C. Poggiale, P. Auger, F. Cordoléani, T. Nguyen-Huu, 2009, Study of a virus-bacteria interaction model in a chemostat: application of geometrical singular perturbation theory, *Philosophical Transactions of the Royal Society - A,* Vol. 367, 4685-4697

•J.-C. Poggiale, M. Gauduchon and P. Auger, 2008, Enrichment Paradox Induced by Spatial Heterogeneity in a Phytoplankton - Zooplankton System, *Mathematical Models of Natural Phenomena*, Vol. 3 (3), 87-102

•P. Auger, R. Bravo de la Parra, J.-C. Poggiale, E. Sanchez, L. Sanz, 2008, Aggregation methods in dynamical systems and applications in population and community dynamics, *Physics of Life Reviews*, 5, 79-105

Application to a fishery problem, by R. Mchich, session MAT03, Room 110, 15:50

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Claude MANTE
Marcos MARVA
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Tri NGUYEN HUU
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Eva SANCHEZ
Richard SEMPERE
Georges STORA
Caroline TOLLA

… and thanks for your attention!

Introduction

• How can we used data got in laboratory experiments to field models? How can we take benefit of the large amount of data obtained at small scales to understand global system functioning?

• Can we link different data sets obtained at different scales?

• Can we envisage a mechanistic approach for ecosystem modelling?

• For a given process in a complex system, what is the effect of its mathematical formulation on the whole dynamics?

Aggregation of variables for systems with several time scales

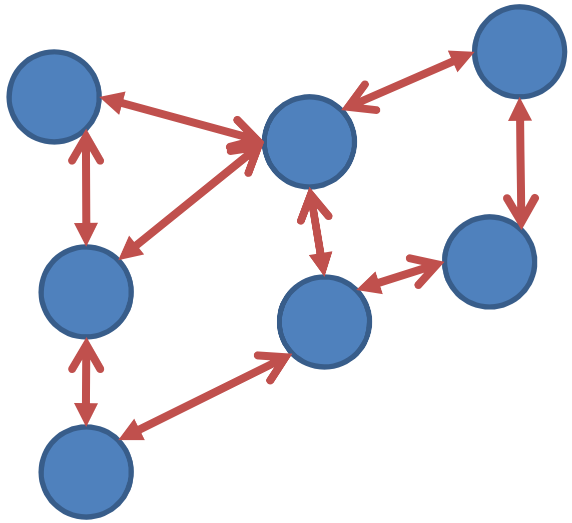
$$(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$$

$$\frac{dx}{dt} = M_x x + \varepsilon F(x, y)$$

$$\frac{dy}{dt} = M_y y + \varepsilon G(x, y)$$

$$\mathbf{1}^T \cdot M_x x = 0, \forall x \in \mathbb{R}^n$$

$$\mathbf{1}^T \cdot M_y y = 0, \forall y \in \mathbb{R}^n$$



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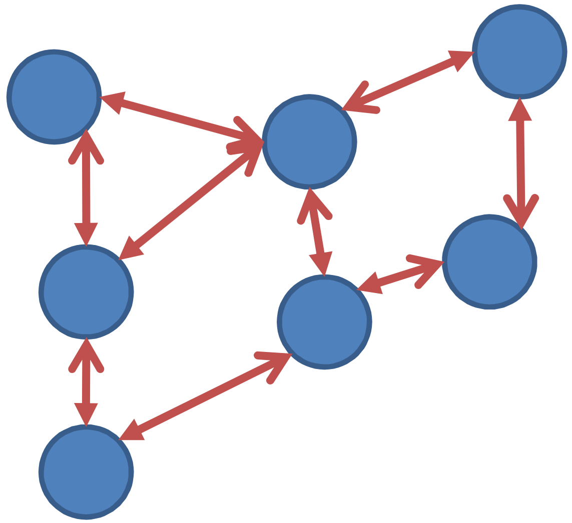

$$\mathbf{1}^T \cdot M_x x = 0, \forall x \in \mathbb{R}^n$$

$$\mathbf{1}^T \cdot M_y y = 0, \forall y \in \mathbb{R}^n$$

$$X = \mathbf{1}^T \cdot x \qquad u = \frac{x}{X}$$

$$Y = \mathbf{1}^T \cdot y \qquad v = \frac{y}{Y}$$

Systems involving several time scales

$$\frac{dX_i}{d\tau} = F_i(X, Y) + \varepsilon f_i(X, Y), \quad i = 1, \dots, k - N, \tag{6a}$$

$$\frac{dY_j}{d\tau} = \varepsilon G_j(X, Y), \quad j = 1, \dots, N, \tag{6b}$$

$$\frac{d\varepsilon}{d\tau} = 0. \tag{6c}$$

(C1) When ε is null in system (6), then Y is a constant. We assume that, for each $Y \in \mathbb{R}^N$, there exists at least one equilibrium $(X = X^*(Y), Y, 0)$, defined by $F_i(X^*(Y), Y) = 0$, $i = 1, \dots, k - N$. We define the set:

$$\mathcal{M}_0 = \{(X, Y, \varepsilon);\ X = X^*(Y);\ \varepsilon = 0\}.$$

This invariant set for the nonperturbed system shall play the role of the invariant normally hyperbolic manifold mentioned in the GSP theory.

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This invariant set for the nonperturbed system shall play the role of the invariant normally hyperbolic manifold mentioned in the GSP theory.

(C2) Let us denote $J(Y)$ the linear part of system (6) around the equilibrium $(X^*(Y), Y, 0)$. We assume that the Jacobian matrix $J(Y)$ has $k-N$ eigenvalues with negative real parts and $N+1$ null eigenvalues. With this condition, the set $\mathcal{M}_0$ is said *normally hyperbolic* since, at each point in $\mathcal{M}_0$, the restriction of the linear part to the $\mathcal{M}_0$ normal space has negative eigenvalues. We now give the statement of the main theorem.

Theorem 1. *Under the conditions* (C1) *and* (C2), *for each compact subset* Ω *in* $\mathbb{R}^N$ *and for each integer* $r > 1$, *there exists a real number* ε_0 *and a* C^r *application* Ψ,

$$\begin{aligned} \Psi : \Omega \times [0; \varepsilon_0] &\to \mathbb{R}^{k-N} \\ (Y, \varepsilon) &\mapsto X = \Psi(Y, \varepsilon) \end{aligned}$$

such that:

(1) $\Psi(Y, 0) = X^*(Y)$;
(2) *the graph* $\mathcal{W}$ *of* Ψ *is invariant under the flow defined by the vector field* (6);
(3) *at each* $(X^*(Y), Y, 0) \in \mathcal{M}_0$, $\mathcal{W}$ *is tangent to the central eigenspace* E^c *associated to the eigenvalues of* $J(Y)$ *with null real parts.*

Fenichel N. Persistence and smoothness of invariant manifolds for flows. Indiana University Mathematics Journal 1971;21:193–226.

Wiggins S. Normally hyperbolic invariant manifolds in dynamical systems. AMS, vol. 105. New York: Springer-Verlag; 1994.